MATH 4573: HOMEWORK 8

INSTRUCTOR: TYLER GENAO

Due: March 27, 2026.

This homework has two sections: the first section has the assigned problems that you will turn in to Gradescope for credit. The second section contains recommended and bonus problems, either from myself, the textbook or other sources. These latter problems are not graded for credit, but you may find them to be useful practice and/or interesting!

For any assigned problem in this homework, **you must show all of your work in order to receive full credit. Your solutions can only cite up to §5.3 of our notes. Everything else must be proven.**

1. Problems to Submit

Exercise 1. Determine whether the following lines have integral solutions. If they do, then give a complete description of them.

a) $L_1 : 10x - 7y = 17$.
b) $L_2 : 903x + 731y = 60$.
c) $L_m : mx + (m+1)y = 10$, where $m > 0$ is a fixed integer.
d) $L_n : (n-1)x + (n+1)y = 4573$, where $n > 1$ is a fixed odd integer.

Exercise 2.

a) Show that the line

$$L : ax + by = c$$

has an integral point if and only if for any $n \in \mathbb{Z}$ the line

$$L_n : ax + by = na + c$$

has an integral point. Briefly argue that this still holds if we replace na with nb in L_n.

b) Use part a) to show that the line

$$L : 100001x + 101001000100001y = 1000010$$

has infinitely many integral solutions.

Exercise 3. Show that the Diophantine equation

$$x^2 + y^2 = 9z + 6$$

has no integral solutions.

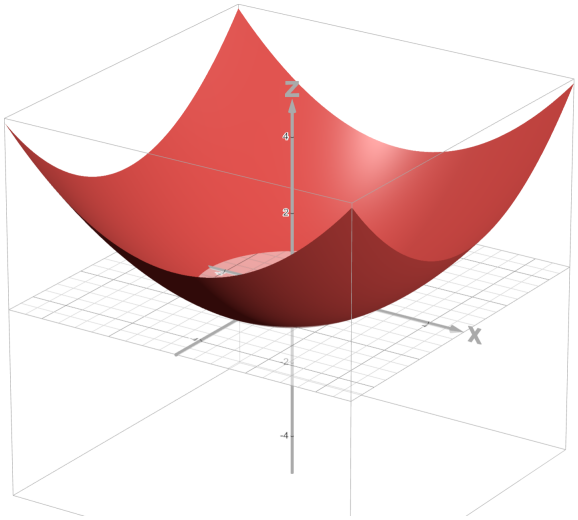

FIGURE 1. The *quadric surface* $x^2 + y^2 = 9z + 6$, pictured in $\mathbb{R}^3$.

Exercise 4. Show that the Diophantine equation

$$x^8 + 1 = 7y$$

has no integral solutions. However, demonstrate that it has infinitely many rational solutions.

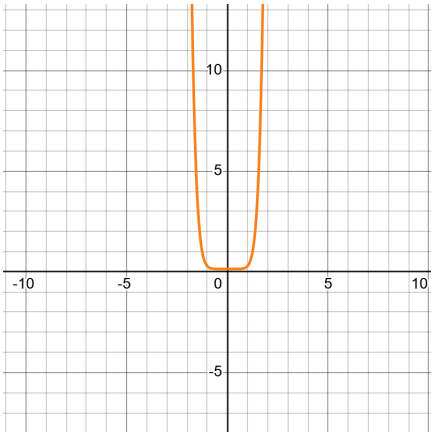

FIGURE 2. The plane curve $C : x^8 + 1 = 7y$, pictured in $\mathbb{R}^2$.

Exercise 5. Show that every Pythagorean triple (x, y, z) is such that 3 divides (at least) one of x, y, z and 5 divides (at least) one of x, y, z.

Exercise 6. Determine all primes p such that the equation

$$x^2 - y^2 = p$$

has integral solutions. For those p which it has integral solutions, determine how many such solutions there are.

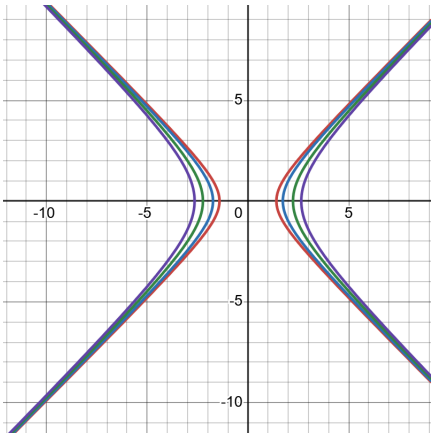

FIGURE 3. The hyperbolas $C_p : x^2 - y^2 = p$ for $p = 2, 3, 5$ and 7, pictured in $\mathbb{R}^2$.

Exercise 7. Who did you consult for this assignment? What resources did you use?

2. Other Recommended Problems

From [NZM91, §5.4], page 239: $\#1 - 2, 5 - 6$.
From [NZM91, §5.1], pages $218 - 219$: $\#2 - 12$.
From [NZM91, §5.3], page 233: $\#1, 4 - 8$.

Bonus Exercise 8. $> 99\%$ of people can't solve this! Find all $\in \mathbb{Z}^+$ with

$$\frac{\quad}{\quad + \quad} + \frac{\quad}{\quad + \quad} + \frac{\quad}{\quad + \quad} = 4.$$

(*Hint:* see https://mathoverflow.net/questions/227713/estimating-the-size-of-solutions-of-a-diophantine-equation.)

Bonus Exercise 9. $99.\overline{9}\%$ of people cannot solve this one! For $\in \mathbb{Z}$ with ≥ 3, find all $\in \mathbb{Z}^+$ with

$$\quad + \quad = \quad .$$

(*Hint:* FLT.)

References

[NZM91] I. Niven, H.S. Zuckerman and H.L. Montgomery, *An introduction to the theory of numbers*, 5th Ed., John Wiley & Sons, Inc., New York (1991).